The theoretical aspects of mathematical modeling in the banking on the example of compound interest

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Abstract

The compound interest is used in banking. It is one of the ways of establishing the interests (the compound interest), so the fee for the right to have the money at the disposal. It is associated with the multiple calculating of the same percentage from the capital whereby every time the interests are capitalized; they increase the base from which the next percentage is calculated. In this paper are discussed the matters concerning the annual, subperiod and continue capitalization and also associated with them the term of percentage: annual, subperiod, nominal, effective, continuing, and average. The reader will find here the mathematical models enabling to calculate:

- the final capital for the initial capital at the given percentage for the fixed time,
- the percentage on the basis of the initial and final capital after the established time,
- the time of percentage on the basis of initial and final capital after the certain time.

The passage of this article concerns the matter of equivalence of the percentage within the time. With the usage of the mathematical models the conditions of equivalence of the percentage were described: the nominal ones (especially subperiod and annual), subperiod and continuing, subperiod and effective, effective and continuing, subperiod and simple percentage.

The relationship between the terms is expressed in the form of the mathematical formulas. The presentations of the chosen models on the diagrams or in the tables will ease the reader to understand these terms.

The content included in this paper will put the reader flesh on the basics of the theoretical banking calculations in the condition of the compound percentage and it will also show how important in the banking mathematic is.

Keywords: compound interest, interests, capitalization of the interests.

Introduction

Bank is the institution which deals with the cash operations. The term interest is associated with its activity. It is the fee for the right to operate the monetary capital calculated as the percentage. The value of the interest is depended on the percentage, the capital size, the time duration on which the capital was made available for and also the rules according to which the calculations are made. The basics of one of them is the presented in this paper the model of calculations called the compound percentage.

Material and methods

The compound percentage is the type of calculating which is characterized by the capitalization of the interest—after the passage of the time of the percentage, the interests are added to the capital from which they were calculated and in the next period of time the

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percentage is calculated from the capital of the increased value.

The factor \((1+r)\) which is in all formulas is called the annual percentage factor.

The equations (25) and (26) create the model of compound percentage within the annual capitalization (also known as the model of annual percentage).

From the mathematical point of view within the compound interest, the future value of \(F\) capital is the exponential function of percentage time.

We are introducing the following indications:
- \(P\) - the initial value of the capital
- \(r\) - the annual interest rate
- \(n\) - the time of the bank rate expressed in years
- \(I\) - percentage
- \(F\) - the future (final) value of the capital after \(n\) time

Then the percentage for a year \(n\) is equal to

\[
I = P (1 + r)^n \cdot r
\]

(1)

and the capital after \(n\) time is expressed by the following formula

\[
F = P (1 + r)^n
\]

(2)

whereas the total interest after \(n\) years of compound interest and the annual capitalization describes the formula

\[
I = P \left[ (1 + r)^n - 1 \right]
\]

(3)

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From the mathematical point of view within the compound interest, the future value of \(F\) capital is the exponential function of percentage time.

Figure 1. The picture presents the initial capital \(P\) and their future values \(F_1, F_2, F_3\) after the further capitalization of the interest.

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Figure 2. The value of the capital as the time function. The marked points on the graph of the function represent the value of the capital within the moment of the capitalization of the interest.

In practice the dynamics of the process of capital changes within the time is crucial. Its mathematical measurements are the absolute and relative increase.

Table 1. The absolute increase and the relative increase of value of the capital within the simple and compound percentage

<table>
<thead>
<tr>
<th></th>
<th>simple percentage</th>
<th>compound percentage</th>
<th>comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>the absolute increase of the value of the capital</td>
<td>(Pr)</td>
<td>(P (1 + r)^n \cdot r)</td>
<td>Within the simple percentage the absolute increase of the capital is stable in the time.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Within the compound interest the absolute increase of the capital is the increasing exponential function of time.</td>
</tr>
<tr>
<td>the relative increase of the capital value</td>
<td>(\frac{Pr}{P(1 + mn)})</td>
<td>(\frac{P (1 + r)^n \cdot r}{P (1 + r)^n - 1} = r)</td>
<td>Within the simple percentage the relevant increase of the capital is the decreasing function of time. In the compound interest the relative increase of the capital is stable in time.</td>
</tr>
</tbody>
</table>
Using the annual percentage model, it is possible to appoint all the values.

<table>
<thead>
<tr>
<th>mathematical model</th>
<th>theoretical counterpart</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculating the final capital $F$ from the initial capital $P$ with the established $r$ interest rate for the $n$ percentage time years.</td>
<td>$F = P(1+r)^n$ Calculating of the value of the exponential function for the particular argument.</td>
</tr>
<tr>
<td>Calculating of the annual interest rate knowing the values of initial $P$ capital and also final $F$ capital and the time of $n$ percentage years.</td>
<td>$r = \sqrt[n]{\frac{F}{P}} - 1$ Calculating of the basis of exponential function.</td>
</tr>
<tr>
<td>Calculating of the time of the percentage of $n$ years knowing the values of the initial $P$ capital and final $F$ capital and annual interest rate $r$.</td>
<td>$n = \frac{\ln \frac{F}{P}}{\ln (1+r)}$ Calculating of the argument for which the exponential function has its particular value.</td>
</tr>
</tbody>
</table>

From the model of the annual compound interest the following conclusions can be made:

- The values of the capital $F_n$, $n = 0, 1, 2, 3, ...$ in this model fulfill the recursive relationship

$$\begin{cases} F_0 = P \\ F_{n+1} = F_n(1+r) \end{cases}$$  \hspace{1cm} (4)

- The sequence $I_n$, which is the reflection of the interest for the subsequent years $n = 0, 1, 2, 3, ...$ is the geometrical sequence in which the first word is equal to $Pr$ and the quotient $1+r$.

- The compound interest is equal to simple interest within the same interest rate $r$ at the end of the first year ($n = 1$). In each subsequent year the compound interest is higher than the simple interest.

Within the annual interest rate $r$ and the annual capitalization of the interest, the capital will double its value in the time.

$$n = \frac{\ln 2}{\ln (1+r)}$$  \hspace{1cm} (5)

In practice it is very common to use the rule 70 according to which the established time of making the capital double can be presented as the following mathematical formula:

$$n = \frac{70}{r}$$  \hspace{1cm} (6)

**The simple percentage as the approximation of the compound interest for the period shorter than the time of the capitalization of interest**

While calculating the interest for the period of time shorter than the period of their capitalization in the bank’s practice, the compound interest is replaced with the simple percentage.

We assume that capital $P$ is yield in $n$ time. The number $n$ is presented in the form of the sum of two components

$$n = m + m'$$  \hspace{1cm} (7)

where $m = \lfloor n \rfloor$ total $n$.

We approve the rule that the interest for the $m$ time is calculated as the compound interest, while for $m'$ - as the simple percentage. Where $m' = 0$, the future $F$ value is calculated using the mathematical formula (2).

When $0 < m' < 1$, calculating the future value of the capital is divided into two stages. First, we calculate the future value of capital $F$ after $m$ periods of compound interest which is

$$F = P(1+r)^m$$  \hspace{1cm} (8)

Next, we determine the approximate value of $F$ capital after $m'$ time of simple percentage which is equal to:

$$\tilde{F} = P(1+r)^m(1+rm')$$  \hspace{1cm} (9)
the future capital \( F \)

\[ F = F \]

\[ F > F \]

\[ F - F \]

Figure 3. The graphs of the functions show how the value of the capital is increased within two variants of the percentage: compound percentage (the fragment of exponential curve) and simple (the fragment of the line) within the shorter capitalization time.

In the case, when \( n \) is the integer number there is the equality \( F = \bar{F} \). If \( n \) is not the integer number \( \bar{F} > F \), from which stems that for the creditor the approximate percentage is more beneficial to the compound one.

**The capitalization of the interest within the subperiods**

In the banking the capitalization of the interest is also done after time shorter than the year. The established time after which the interest is capitalized is named the subperiod of capitalization while the percentage corresponding the subperiod - subperiod percentage. The number which represents how many times within the year the interest is capitalized is called the frequency of capitalization.

We introduce the following indications:

- \( P \) - the initial value of the capital
- \( F \) - the future (final) value of the capital
- \( n \) - the time of the percentage in years constructed from the total number of subperiods
- \( k \) - the frequency of capitalization within the year 
  where \( k = 2 \), the capitalization is semiannual 
  where \( k = 4 \), the capitalization is quarterly 
  where \( k = 12 \), the capitalization is monthly 
- \( i_k \) - the subperiod percentage
- \( m_k \) - the time of the percentage expressed in subperiods, therefore:
  \[ m_k = nk \] (10)
- \( r_k \) - the nominal percentage that is rate of interest within the \( k \)-repeated capitalization of the interest within the year proportional to the subperiod percentage \( i_k \).

Therefore:

\[ r_k = ki_k \] (11)

According to the indications the final capital after \( m_k \) subperiods time is equal to:

\[ F = P (1 + i_k)^{m_k} \] (12)

while combined interest \( I \) after \( m_k \) time amounts to

\[ I = P \left[ (1 + i_k)^{m_k} - 1 \right] \] (13)

Equations (12) and (13) are named the model of compound percentage within the subperiod capitalization (the model of subperiod percentage).

After taking into account the equality (10) and (11) the formula (12) and (13) take the form:

\[ F = P \left( 1 + \frac{k}{n} \right)^{nk} \] (14)

and

\[ I = P \left[ (1 + \frac{k}{n})^{nk} - 1 \right] \] (15)

From the equations (14) and (15) we deduce that within the fixe nominal percentage \( r_k \), the frequency \( k \) of the capitalized interest will be increased (the length of the subperiod will be shortened), the value of the interest and the final capital will be increasing. Therefore, it can be said that the shortening of the capitalization subperiod increases the pace of growth of the final capital. (Bijak, Podgórska, Utkin, 1994).
If the \( F_n \) symbol is the final capital after \( n \) time, and its value after the passage of another year is indicated with \( F_{n+1} \), therefore we can write:

\[
F_n = P\left(1 + \frac{r}{k}\right)^{nk} \quad \text{and} \quad F_{n+1} = P\left(1 + \frac{r}{k}\right)^{(n+1)k}
\]

(16)

The relationship of the value of the capital from the end of any year to the value from the beginning of the year is named the annual factor of the percentage and is indicated with \( \rho_k \) symbol. It has the fixed value within the percentage time (Podgórska, Klimkowska, 2005), which is equal to:

\[
\rho_k = \left(1 + \frac{r}{k}\right) = \left(1 + i_k\right)^k
\]

(17)

Especially the annual percentage factor within the annual capitalization \( k = 1 \) has the value:

\[
\rho_1 = 1 + r
\]

(18)

If the time of the percentage within the subperiod capitalization is equal to \( n \) years, the value of the capital within every year increases \( \rho_k^n \) - multiple. At the end of \( n \) year its value will \( \rho_k^n \) - multiple higher from the value of the initial capital.

Therefore, the value of the final capital can be described as the following mathematical formula:

\[
F = P\rho_k^n
\]

(19)

On the basis of this formula, we deduce that the increase of the annual percentage \( \rho_k \) factor increases the pace of growth of the final capital.

Within the fixed nominal percentage, the annual percentage factor is the higher than the shorter time of the capitalization of interest is. In practice, \( \rho_k \) factor is applied as the measure of the pace of the increase of the capital in the condition of the subperiod capitalization.

The continuing capitalization within the compound interest

The issue of the continuing capitalization concerns the problem how the frequency of the capitalization can be increased in order to, at the given nominal \( r_c \) percentage obtain the highest possible interest.

We analyse the existence of the annual \( \rho_k \) percentage factor within \( k \rightarrow +\infty \).

\[
\lim_{k \to +\infty} \rho_k = \lim_{k \to +\infty} \left(1 + \frac{r}{k}\right)^k = e^r
\]

(20)

where \( e \) is the fixed mathematical named the number of Eulera (Fihtenholz, 1985), that is:

\[
e = \lim_{k \to +\infty} \left(1 + \frac{1}{k}\right)^k
\]

(20)

In the situation when the frequency of the capitalization is ureservely increasing \( k \rightarrow +\infty \) it is said about the continuing capitalization of the interest (continuing percentage). The nominal percentage \( r_c \) which is the equivalent of this capitalization is named the nominal continuing percentage.

In the condition of the continuing percentage within the continuing percentage \( r_c \), the initial capital \( P \) after \( n \) years is increasing to the final value:

\[
F = Pe^{r_cn}
\]

(21)

and it generates the interests:

\[
I = P\left(e^{r_cn} - 1\right)
\]

(22)

The equations (21) and (22) create the model of compound percentage within the continuing capitalization (all known as the model of the continuing percentage).

The formula

\[
\rho_c = e^{r_c}
\]

(23)

describes the annual \( \rho_c \) percentage factor. After taking it into the account, the formula (21) and (22) are of the following:

\[
F = P\rho_c^n
\]

(24)

and

\[
I = P\left(\rho_c^n - 1\right)
\]

(25)

The model of the continuing percentage enables to calculate the continuous capital and the interest for the time of any length. It eliminates the approximation of the compound interest with the simple percentage within the period of time shorter than the time of capitalization.

Within the same nominal percentage the continuous capitalization provides the faster increase of the capitalization than the compound capitalization (Podgórska,
Klimkowska, 2005). For the any \( k \) value it is of the following:

\[ \rho_k < \rho_c \] (26)

which enables to appoint the nominal percentage \( r_k \), which is equivalent to the nominal percentage \( r_{ki} \).

In the special case, the annual \( r \) percentage equivalent to \( i_k \) percentage describes the following mathematical formula:

\[ r = (1 + i_k)^{\frac{1}{n}} - 1 \] (32)

while the \( i_k \) percentage which is equivalent to \( r \) percentage describes the following mathematical formula:

\[ i_k = (1 + r)^{\frac{1}{n}} - 1 \] (33)

From the mathematical formulas (14) and (22) fall out that the value of \( F^{(i)} \) capital with the subperiod percentage and \( F^{(c)} \) with the continuous percentage are respectively equal.

\[ F^{(i)} = P \left(1 + \frac{r}{k}\right)^{nk} \] and \[ F^{(c)} = P e^{r_n} \]

Comparing these values we obtain:

\[ \rho_i = \rho_c \] (34)

which means that the condition of the equivalence of the compound percentage is the equality of the corresponding them annual percentage factors.

The mathematical formula:

\[ i_k = e^{\frac{r}{n}} - 1 \] (35)

enables to replace the continuous percentage with its equivalent subperiod percentage. Whereas the relationship between the continuous percentage to the subperiod percentage determines the formula:

\[ r_c = k \ln (1 + i_k) \] (36)

The basics for the determining of the equivalence of the compound percentage and simple one is the comparison of capital \( F^{(i)} \) with the compound percentage of the interest rate \( i_k \) in \( n \) time:

\[ F^{(i)} = P \left(1 + i_k\right)^{nk} \]

and the capital \( F^{(c)} \) with the simple percentage of the interest rate \( r \) in \( n \) time:

\[ F^{(c)} = P (1 + r)^n \]

Therefore:

\[ \left(1 + i_k\right)^{nk} = (1 + nr) \] (37)
The obtained equality is the condition of the equivalence of the interest rate \( i_k \) and \( r \) within \( n \) time.

The effective interest rate

The annual interest rate equivalent to the compound interest rate is the effective interest rate. It means how much the percentage increases the value of the capital within the year. In the percentage given of the effective interest rate \( r_{ef} \), the annual percentage factor is equal \( 1 + r_{ef} \). In the subperiod percentage of certain interest rate \( i_k \) is \((1+i_k)^k\). Comparing the both factors we got:

\[
r_{ef} = (1+i_k)^k - 1
\]  

(38)

The mathematical formula for calculating of the effect interest rate within the usage of nominal interest rate is of the following:

\[
r_{ef} = \left(1 + \frac{r}{k}\right)^k - 1
\]  

(39)

whereas within the usage of the annual percentage factor:

\[
r_{ef} = \rho - 1
\]  

(40)

In the case when \( k = 1 \) there are the equivalences:

\[
r_{ef} = i = r
\]  

(41)

which means that within the annual capitalization the effective interest rate is equal to the nominal percentage.

The relationship between the interest rate \( r_c \) of the continuous percentage and the effective percentage \( r_{ef} \) determines the mathematical formula:

\[
r_{ef} = e^r - 1
\]  

(42)

The obtained model written with the usage of the annual percentage factor has got the form:

\[
r_{ef} = \rho - 1
\]  

(43)

At the stated nominal percentage:

- the effective and nominal percentage are equal only within the annual capitalization;
- the effective percentage is higher than the nominal one, when the capitalization period is shorter than the year;
- the effective percentage is higher the more often the interest rate is capitalized;
- the effective percentage is higher within the continue capitalization.

The average interest rate

The average interest rate of the capital \( P \) within \( n \) time is called the annual percentage \( \bar{r} \), within which the capital \( P \) generates in \( n \) time the interest rate equal to the interest rate within the differentiation of the percentage in the particular periods.

If the capital \( P \) yield in \( n \) years time, whereby the effective percentage in the next years is \( r_{ef}^{(1)}, r_{ef}^{(2)}, \ldots, r_{ef}^{(n)} \), the terminal value of the capital and the interest rate after \( n \) time, express the mathematical formulas:

\[
F = P \prod_{j=1}^{n} (1 + r_{ef}^{(j)})
\]

(44)

\[
I = P \left[ \prod_{j=1}^{n} (1 + r_{ef}^{(j)}) - 1 \right]
\]

(45)

The equations (42) and (43) present the model of annual interest rate within the variable percentage in time.

The average percentage \( \bar{r} \) is equal (Podgórska, Klimkowska, 2005):

\[
\bar{r} = \sqrt[n]{\prod_{j=1}^{n} (1 + r_{ef}^{(j)})} - 1
\]

(46)

Because \( 1 + \bar{r} \) is the annual interest factor in the condition of annual interest at the fixed percentage \( \bar{r} \) while \( \sqrt[n]{\prod_{j=1}^{n} (1 + r_{ef}^{(j)})} \) is the geometrical average of annual interest factors in the next \( n \) years of the interest, it can be concluded that in the condition of the annual interest within the changeable percentage, the average interest factor is the geometrical average of annual interest factors in this \( n \) time.

The formulas to compute the average subperiod percentage \( \bar{r}_k \) equivalent to the average annual percentage \( \bar{r} \) are of the following:

\[
\bar{r}_k = \sqrt{1 + \bar{r} - 1}
\]

- the average half-yearly percentage

(47)

\[
\bar{r}_k = \sqrt[4]{1 + \bar{r} - 1}
\]

- the average quarter percentage

(48)
\[ \overline{r}_{12} = \sqrt[12]{1 + \overline{r}} - 1 \] - the average month long percentage

The final value of the capital after the passage of m subperiods describes the following model:

\[ F = P \prod_{j=1}^{m} \left( 1 + \overline{r}_{k}^{(j)} \right) \]  \hspace{1cm} (50)

whereby the average percentage is equal:

\[ \overline{r}_{k} = \sqrt[m]{\prod_{j=1}^{m} \left( 1 + \overline{r}_{k}^{(j)} \right)} - 1 \]  \hspace{1cm} (51)
where \( \overline{r}_{k}^{(1)}, \overline{r}_{k}^{(2)}, \ldots, \overline{r}_{k}^{(m)} \) are the percentages in the progressive subperiods \( j = 1, 2, \ldots, m \).

Furthermore the annual interest factor \( \rho \) and the effective percentage \( r_{ef} \) amount to

\[ \rho = \prod_{j=1}^{k} \left( 1 + \overline{r}_{k}^{(j)} \right) \]  \hspace{1cm} (52)
\[ r_{ef} = \prod_{j=1}^{k} \left( 1 + \overline{r}_{k}^{(j)} \right) - 1 \]  \hspace{1cm} (53)

**Results and discussion**

In the condition of the compound interest the future value of the capital is the growing exponential function of percentage time. Therefore, the following conclusions can be made:

1. higher percentage generates higher interest
2. shortening the capitalization time (therefore increasing its frequency) increases the pace of the capital growth;
3. the highest increase of the value of the capital gives the continue capitalization;
4. at the same percentage the equality of simple and compound interest is for the time equal to one period of capitalization, for the longer period of time the compound interest is higher than the simple one.

According to the rule of compound interest the term deposits are yield. The client makes the contract with the bank concerning the entrusted money for the set time. The conditions of yielding the deposits are determined by two parameters- the nominal interest rate and the frequency of the capitalization of the interest. The presented mathematical models enable the possibility:

- to calculate the amount for which the paid-in capital for the deposit will gain after the certain time,
- to calculate the amount which has to be paid-in on the deposit in order to generate the expected final capital after the fixed time,
- to calculate the time of the percentage, in order to, the paid-in money for the deposit, bring the expected profit,
- to calculate the value of the percentage on the basis of the initial and final capital after the appointed time.

The banks offer the term deposits and they give their percentage in a year. Knowing the mathematical tools enable in the case of when the time of the capitalization of the interest is different than a year, to calculate the effective percentage and to establish how much the paid-in amount of money increases in reality.

Making the deposits is the form of investing the money. The banks present different offers. Thanks to the mathematical
models it is possible to assess their profitabilities. The basics of the comparison of the conditions of the percentage can be the calculating of the effective interest rate or calculating of the equivalent percentage in time within different capitalization periods. Banking transactions made enable to choose the most profitable offers.

Conclusions

The client of the bank who deposits her, his savings becomes its creditor. Of this title, he or she gains the profit in the form of the interest which value is established with the usage of the mathematical instruments. The better knowledge of the mathematics enables to make better financial decisions.

Without the basic mathematical knowledge, it is not possible to understand complicated banking matters. The acquaintance of the terms and the models presented in this paper gives the chance of the efficient functioning in the condition of the compound percentage.

References


